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Agreeing to disagree under a Quantum-like decision framework:

Implications for costly signalling in economic science

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1. Introduction

Last decade has witnessed a significant upsurge in the research in the area of Quantum-like modelling (Bagarello et al, 2018) in social sciences. There is a growing body of literature (Haven et al, 2017) which provides both theoretical and empirical justifications of reformulating decision-making theory based on the mathematical and logical formulation of Quantum theory. Authors (Haven and Khrennikov, 2017) have formulated decision-making models and found empirical support of the predictions (Pothos and Busemeyer, 2013) in cognitive psychology studies, or in real life financial market, or general voting scenarios. However, application or extension of such quantum-like models in financial economics is still in a nascent stage (Haven and Khrennikova, 2018, Khrennikova and Patra, 2019).

Haven and Khrennikov (2013), Khrennikov (2015) have formulated a general framework of 'common knowledge' among agents based on Quantum-like modelling. Common knowledge formation is the central theory for Economics, mainly financial economics. The behaviour of rational agents in any kind of equilibrium model (Bar-Issac, 2003) is based on the premise of common knowledge formation at the equilibrium. However, Quantum-like formulation of common knowledge has challenged and generalized the central Aumann (1976) theorem. Hence important questions arise, for example, how this new formulation can be extended to agent-based decision-making theory in Economics? Mainly when we try to explain decision making under uncertainty? In mainstream Neoclassical theory, costly signalling models (Aerlof, 1972, Spence 1979, Bhattacharya,1979) have been proposed to explain agent's behaviour in such scenarios. However as already noted in numerous studies (Haven and Khrennikov, 2013) that the standard model fails to describe behaviour under true randomness or deep uncertainty[1].

Hence the current paper is an early attempt to build a costly signalling model under uncertainty based on Quantum-like formulation of common knowledge. The paper also discusses the contrasting features of standard Bayesian probability updating model. Overall the Quantum-like model is a more general model where under special conditions typical signalling solution can be achieved even if the agents update their belief states based on Quantum probability rule.

Costly signalling literature developed in the 1970s mainly for resolving information asymmetry problems, for example, the widely researched adverse selection problems in different markets, eg, job market, insurance market, credit markets, equity market and other related areas. Recent

[1] In standard Financial Economics risky situations based on Bayesian probability updating models are rather studied as uncertain situations, in Quantum theory however, the deep uncertainty is not simply based on epistemology or lack of knowledge, but a fundamental feature of reality, or ontological.

papers (Bar-Issac, 2003) emphasize that such signalling models can be extended to many other scenarios, for example, commodity markets.

The central problem remains information asymmetry, where one or some parties in a transaction or a contract have less than perfect information, or less information than the other parties, on different features, for example, the quality of products, or a firm's future profitability or investment profile, and alike issues. In such a scenario rational agent may perceive markets to be occupied by different categories of firms or producers, for example, high and low types, or further finer discrete or continuous spectrum. Again in such a scenario, the genuine firms would find in its own utility maximization behaviour to send a sufficiently costly[2] signal so that rational agents can update their beliefs regarding the quality of firms, and a separating equilibrium is created such that agents can screen one type from the other, and price them accordingly in markets[3][4], Hence solving the embedded adverse selection problem.

In the standard signalling literature in financial economics (Akerlof, 1972, Spence 1979, Bhattacharya,1979) fewer studies have been done on how rational agents learn about the true nature of the signals, it is always assumed that the agents are 'Bayesian' rational, or they start with a common prior belief and update according to Bayesian rule and form a common posterior belief based on the possibility of common knowledge formation, hence creating a separating equilibrium. Thus Aumann's (1976) 'not agreeing to disagree' theorem is a central assumption, which is or at times is not explicitly mentioned[5]. However, there are some critical issues which are not resolved, and these issues are both theoretical and practical so to say.

Theoretical issue is with the nature of uncertainty and the rule of belief updating, and the practical issue is the inability of the standard signalling theory to explain divergent beliefs or behaviour of rational agents around costly signals (or behaviour of agents given noisy signals). Costly signalling theory is based on the classical set theory or measure theory of probability where there is no inherent randomness or uncertainty, or in other words, it's theoretically possible to gain perfect knowledge about a system[6] . However, such a theory is ill-equipped to explain true uncertainty which agents face in the market scenarios. Hence quantum-like modelling is based on deeper or true randomness which is described as a pure state or superposition of orthogonal basis states[7], which can describe an initial belief state or a common prior as in the minds of the agents[8].

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[2] Costly is the key word here, since only a costly activity like dividend payouts can create a signal here since it would be difficult for the less genuine firm to copy that activity.

[3] Which also mean price discovery here.

[4] Bayesian signalling models have been developed to predict price formation, or, price mechanisms. However, since here, we are dealing with a Non-Bayesian belief updating model the inherent price mechanism is different.

[5] Certainly exceptions are noted early:

[6] Say initial conditions of a system

[7] We assume an abstract Hilbert space (finite dimensional in a simple case) spanned by orthogonal basis states as the state space in quantum-like modelling. There are quite a few advantages of doing this as is described in the subsequent sections.

[8] In standard financial economics, or decision theory, risky scenarios can be described by assigning probabilities to probable world states, which is the standard ensemble view. However, uncertainty state, which is the most

Financial economics literature covers private information issues in detail (**Tirole,2010**). Modern asset pricing literature is based on the problems of adverse selection, and, or moral hazard. Equity premium puzzle is based on the assertion that there is a presence of information asymmetry between the less informed and more informed agents which has a significant impact on the equity returns. Dispersion of investors beliefs Miller, 1977, Chatterjee et al, 2015) is another strong strand of literature which is related to private information. Even the capital structure of firms is significantly affected by the degree of information asymmetry in the market. Here again, we observe that deep uncertainty in financial markets, which is beyond simple asymmetry of information, is not fruitfully described in standard signalling theory.

The practical problem, or the empirical problem generates from the fact that real choice data (for example in the scenario of signalling by firms: investing or not investing in assets) under uncertainty demonstrates some 'contextual' or deviant features (Haven and Khrennikov, 2009) which cannot be explained by the Bayesian rationality model. Specifically, in the case of signalling phenomena, the existence of divergence of opinions of agents (shareholders, or analysts via analyst forecasts) cannot be explained by the central Aumann theorem. Such divergence of opinions has a significant impact on asset prices (Chatterjee et al,2012) which may deviate such values way beyond so-called fair price, hence undermining the very purpose of costly signalling. We observe here that in standard theory there is no consensus on the explanation of divergence of opinions, which make the asset prices deviate from the equilibrium or fair price.

No doubt there has been a strong response from the behavioural finance school to explain such divergent behaviours, but here we just remark that the diverse behavioural school is at times less coherent, and the explanations are mainly based on heuristics, thumb rules and same Bayesian learning models (Haven and Khrennikov,2013). Hence the emerging quantum-like paradigm (Haven, Khrennikov and Robinson,2017) attempts to formulate a more general probability updating or decision-making theory with deep consequences for Aumann's theorem or alike propositions.

The proposed model is an early attempt to build a more general model of cognition or agents' behaviour under uncertainty in markets rather than typical information asymmetry scenario, where agents may or may not continue to agree on the posterior probabilities even if they start from a common prior and common knowledge is guaranteed about posterior beliefs. Hence the typical costly signalling equilibrium behaviour by the rational agents can be challenged.

In the proposed model (which adapts the framework of a standard signalling model, for example, Bar-Issac 2003) we have a monopoly risk neutral seller and risk-neutral buyers. Seller would like to maximize the discounted present value of profits (with a discount rate $\beta \epsilon$ (0,1)), and at the beginning of every period, she decides whether to sell or not. On the other hand, the buyers have a common prior belief about the quality of the seller, which is termed as the

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important state in quantum theory, is a superposition of all possible states which is not an ensemble picture and often called the pure state. One can have a more general representation, the density matrix, which is the ensemble of many pure states.

reputation of the seller. Belief updates happen based on information revelation, and such belief updates help the seller to decide whether to continue trading for the next period, and the buyers to revise the common belief about the reputation.

We show that there are multiple ways in which the results from this model differs if the underlying decision making/belief update mechanism is Quantum-like rather than standard Bayesian. For example, say in the scenario where the buyers and the sellers are ‘uninformed' about the type of products or no party has a greater information (i.e. the case of symmetric information), it can always happen for Bayesian rational agents that if the common prior is near 0 it remains near 0 irrespective of any belief update. Hence even a good seller can be punished if the common prior is very low and has to leave the market. On the other hand, if the agents update beliefs based on the Quantum-like formulation there is no such zero-prior trap (), and a genuine seller even with a low prior reputation can sustain in the market.

Specifically, we start with a standard framework of agents or decision makers in a market and a seller (may be of any financial asset, or any good). There can be three scenarios, one, where both the seller and the agents are equally uncertain about the success of the good/ asset, where the only prior available to the agents is a prior belief about the reputation of the seller, the second scenario is where the seller is fully knowledgeable about the quality of the good/asset whereas the agents are not, hence there is a possibility of costly signalling, the third scenario is when the seller is only partially knowledgeable about the quality.

Our assumption is that the agents are quantum-like or uses Born's rule to update probabilities. The main aim is to investigate whether Aumann's common knowledge theorem will still hold. If and only if the theorem still holds there can be a costly signalling equilibrium, where the market would update the posterior probabilities which would then be reflected in prices.

Hence Quantum-like belief updating model is socially more efficient since in standard Bayesian paradigm due to zero prior trap/ Cromwell's rule good quality sellers may be doomed and society must put up with inferior quality goods, at least in this scenario of uninformed agents to begin with.

The other big difference is certainly about the formation of the common posterior belief itself, it is not necessary for the Quantum-like model that agents must agree, or they can continue disagree even if the posterior probabilities are known.

In the classical common knowledge theory, we begin with the famous theorem by Aumann (1976) which states that rational agents when starting with a common prior belief state cannot forever disagree on posterior belief states / posterior probabilities if there is common knowledge about the posteriors.

Mutual knowledge can be defined as a state where everyone in a group knows about a specific event. *Common knowledge* is rather a stricter version where Alice and Bob know a specific event E, again, Alice knows that Bob knows the event E, again... hence there are many orders of common knowledge (Haven et al, 2017).

To repeat, then, Aumann's theorem states rational agents[9] starting with common prior about an event if finds that posteriors about that event are a common knowledge then posteriors must be equal, they just cannot agree to disagree.

Here certainly by rational, it is meant that agents are Bayesian rational, or in other words, they update belief states based on the Bayesian probability update scheme. Aumann's theorem has been an integral part of rational decision-making theory, specifically, in game theory, asset pricing models, rather any standard financial market modelling. For example, the commonly used CAPM model has the basic assumption of homogenous expectations of rational agents in equilibrium, which is based on the common knowledge theorem. Here we show that common knowledge can be conceptualized both from the perspectives of classical set theory (or Kolgomorovian measure theory) and quantum logic[10].

Deductive and inductive reasoning

Another supposed contribution of quantum like modelling of common knowledge is that we necessarily don't need to assume that agents are acting based on complex inductive reasoning. Behavioural finance, as well as the emerging theory of complexity economics or finance (Arthur, 2015 for example), do place a central importance on inductive reasoning, where agents who are bounded rational update their beliefs and hypotheses about the real world as more information is disclosed. Such inductive reasoning also calls for heuristics and at times abandon any kind of probability-based decision-making theory altogether.

However, one can hold based on quantum like modelling that rational agents who update beliefs according to quantum decision theory rules, can still remain committed to deductive reasoning and rationally disagree on the outcomes, for example asset prices. Hence in the current paper, as one of the main outcomes of the modified common knowledge model, we will have asset price dispersion rather than homogenous expectations generating single equilibrium asset price. However, for such a dispersion we need not hold agent's irrational or inductive traders, so to speak.

The remaining sections of the paper describes the proposed set up, and also provides examples from asset pricing or signalling literature. Appendix section provides some basic mathematical discussions.

2. Proposed model

[9] Where rationality means every agent update beliefs according to Bayesian rule, and this procedure is also common knowledge.

2.1 Basic set up

There are two types of sellers, good and bad, the probability of success of the good seller is g, and the probability of success of the bad seller is b, s.t. 1>g>b>0. For the risk-neutral buyers the discount rate is r. The common prior probability of the seller being good is λ_t, hence of being bad is $(1-\lambda_t)$, this is the reputation to start with[11]. The buyers assign a value 1 to the success and 0 to a failure, hence the ex-ante value of buying from a good seller is g, and that from a bad seller is b, if the cost of production is c, then we assume g>c>b, which means that transaction with a good type seller is a socially efficient and with a bad of seller is socially inefficient. The price which emerges out from each period is just the valuation of the buyers for that product in the period (based on the assumption that buyers are Bertrand competing).

The costly signalling event which the buyers observe is that decision of the seller to continue selling for the next period, which revises the common belief to say μ_t, which means the price at which trade occurs is $\mu_t g+(1-\mu_t)$ b.

In the standard model set up () there are two belief state updates, one where the costly signal is the decision of the seller to continue for the next period, and two, when the buyers observe the success or the failure of the product after buying/ consuming, which further updates the belief to a final state (i.e. of the reputation). Nature of the equilibrium is thus Markov perfect equilibrium.

Again in standard set up where the agents are Bayesian rational, we can imagine different independent scenarios with their own specific outcomes: scenario1 where the buyers and the seller are equally informed and unsure about the quality, in such a scenario if the prior Λ_t is less than a threshold value then irrespective of the true nature of the seller she has to abandon the market. Scenario2: where the seller is perfectly informed about her type, and the buyers are ignorant, however, the later can update belief based on the Bayesian formulation if they find the seller decides to sell for the next time (even after taking some loss in the first period). All other scenarios are in between these two extreme states, for example, partial private information held by the seller.

Remark 1, scenario one: Quantum-like formulation of belief updating: based on the brief background of Hilbert space modelling and operator representation of observables we propose the following.

Observables can be represented by Hermitian operators, and[12] here we consider two such observables A and B, where B is the observable for the seller in the model being good or bad, i.e. the reputation of the seller, and A is an observable related to any other information:

[11] It is this belief about reputation which is further reflective of the success probability of the product which gets revised over periods.

[12] Hermitian property or the self adjoint property of an operator may not be relevant in human decision-making model, recently authors have suggested POVM or positive operator value measures as an alternative. A short introduction is provided in the appendix.

B = $\{\theta\}$, and A=$\{X\}$, again if B and A are given Hermitian operator/projection operator representation on the relevant Hilbert space then based on the spectral decomposition we have

B = $\sum_{\theta} \theta P$ and A = $\sum_{X} XM$, where P and M are the projection operators.

In the case of the Bayesian model, agents assign a conditional probability to the reputation of the seller:

$$\pi(\theta;\ X) = \frac{P(X;\theta)P(\theta)}{P(X)}$$

Where P(θ)is the common prior probability for good reputation in our model which is Λ_t, again it is clear that if Λ_t is not significantly different from 0, the posterior or the conditional probability $\pi(\theta;\ X) = \frac{P(X;\theta)P(\theta)}{P(X)}$ is also not significantly different from 0.

For Quantum-like model we can first prepare the belief state of the agent, say in this case that of a buyer. The buyer has an initial pure state of belief Iψ> which can be updated when faced with a question for the observable A; AIψ> updates the state to Iψ^1>, and then the agent faces a subsequent question related to the observable B (here about the reputation of the seller), i.e. BIψ^1>, hence here too we compute the conditional probability$\pi(\theta; X)$, however, the formula becomes

$$\pi(\theta; X) =< \psi^1, P, \psi^1 >/ \frac{1}{\|M\psi\|^2}$$

The above formula then is not necessarily tied to the zero prior traps.

However as mentioned earlier if A and B observables are compatible with each other or $[A, B] = 0$ then Bayesian probability updating and Quantum probability updating results are same.

Remark 2: Mixed state representation

In standard Neoclassical Economics/Finance theory () a representative agent is modelled, which is equivalent to proposing that equilibrium agents hold homogenous beliefs (). In Quantum-like modelling we can be more realistic by starting with a mixed state, of say the market. For example in the current model we can assume that ϱ is the mixed state of the market as a whole, which is an ensemble of pure belief states of individual agents, ϱ=$\sum p_i$ Iψ><ΨI = $\sum p_i P_i$, where p' s are the probabilities or frequency measures and P' s are the orthogonal projection measures. Here we assume that individual belief states are independent, rather than entangled. Trace (ϱ)=1.

We can imagine preparing the state of the market in a similar way as earlier for the individual agent, i.e. in two subsequent steps, first updated by A operator and then by B operator.

$P(A=a_i/\varrho) = Trace(\varrho\ P_i)$

Subsequently, ϱ gets updated to ϱ^1 when the market state confronts with B, hence

$P(B=b_j/A=a_i, \varrho) = Trace(\varrho^1 M_j)$, Whereas earlier in our model MJ is the orthonormal projectors for the observable B.

A critical note here is that if the belief states of the agents are entangled then the market state is much more complex.

Posterior belief formulation in Quantum-like modelling: implication for market price

Remark 3, Scenario 2: Again in the standard model () if we are in scenario 2 where the seller is perfectly informed about her type, but buyers are ignorant and only have a common prior Λ_t to start with, then it is required of the seller to send a costly signal (for example the decision to sell in the next period also even having taken some loss in the first period, which is costly, and too costly for a bad type seller) to the market, which may create a so-called separating equilibrium (), which requires buyers to update the probability of good type to μ_t. Here certainly in a standard asset pricing set up μ_t is common knowledge and is the same for all buyers.

In the Quantum-like model we assume there is a seller (the good or bad type) and at least two buyers/agents, i=1,2 who are faced with the dichotomous question – observables, $A^{(i)}$, taking values 0, 1. The state spaces of the agents are considered as to be 2 D Hilbert spaces, H_1 and H_2, Eigenvectors corresponding to Eigenvalues 0,1 are I0> and I1> respectively. For the specific model we assume that I0> means the belief state that the seller is of good type, and I1> represents the belief state of an agent that the seller is of the bad type.

Quantum state update: projection postulate

Generally if we have a given state ρ, and one observable A (which can be represented by a self-adjoint operator in specific and a positive semi-definite operator in general) such that, $A=\sum_i^n aiPi$, where a's are the Eigen values and Ps the orthogonal set of projectors, then probability of realizing a specific Eigenvalue is given by the Born's rule as $p(a_i)=Trace(\rho P_i)$[13].

We then consider the Tensor product of the state spaces of the two agents[14], where the canonical basis of the composite space has the for Imn>, where m,n = 0,1. We consider two orthogonal decompositions of H: $H=H_0^{(1)}+H_1^{(1)}$ and $H=H_0^{(2)}+H_1^{(2)}$, where the subspaces have the basis (I00>, I01>) for H_0^1, (I10>, I11>) for $H_1^{(1)}$, and (I00>,I10>) for $H_0^{(2)}$ and (I01>,I11>) for $H_1^{(2)}$. We have the corresponding projectors denoted as $P_j^{(i)}$, I = 1,2 and j = 0,1, where $P_0^{(i)}+P_1^{(i)}=I$, hence we have two spectral families $P^{(i)} = \{P_0^{(i)}, P_1^{(i)})$ which forms the information

[13] For observing another variable immediately after observing A, we need to follow the sequential projection procedure, such that we get a conditional probability version of Quantum theory.

[14] We consider here the market as a combined system, though may not be a typically entangled system which can never be decomposed into product spaces (in Quantum theory these are called as maximally entangled Bell states).

representations for agents, 1,2. Again the spectral families are compatible since the projections corresponding to different agents commute.

We first consider a common prior state ϱ, which is a deep uncertainty state, which means that ϱ is the density matrix representation of the pure state. In this case, we set the pure state $\Phi = C_{00}I00>+C_{01}I01>+C_{10}I10>+C_{11}I11>$, where $\sum C_{nm}^2 = 1$ and $C_{nm} \epsilon R$.

This description is already richer than the standard Bayesian set up where we start with a common belief state of Λ_t, as defined earlier, assuming that agents are homogenous or representative, here from the start we have agent heterogeneity.

Here $\varrho = I\Phi><\Phi I$.

Now we define an Event as a signal sent by the seller, say the signal is the decision to continue selling in the next period. Corresponding to this we compute the prior and posterior probabilities or, the prior and posterior beliefs of the agents (or the market as a whole) that whether the event is revealing the type of the seller.

This event E can be given a maximal uncertainty description, say $I\psi> = 1/2[I00>+I01>+I10>+I11>]$, hence the prior probability for this event to happen is $q(E)=II<\Phi,\psi>II^2 = ¼[C_{00}+C_{01}+C_{10}+C_{11}]$.

However, the agents now update the state Φ according to their information representations ($A^{(i)}$, or the spectral families as described earlier)(a more detailed description of Quantum modelling of common knowledge formation is discussed in the Appendix1). Here the state updating of the agents are based on the standard Luder's projection rules where pure states are updates to pure states, hence we can proceed with the following pure states only.

Based on the updating rule as described earlier, for the first agent, we have Φ updated to

$\Phi_0^1 = (C_{00}I00>+C_{01}I01>)/[IC_{00}^2+C_{01}^2]^{1/2}$, or, $\Phi_1^1=(C_{10}I10>+C_{11}I11>)/[C_{10}^2+C_{11}^2]^{1/2}$, similarly for the second agent we have states Φ_0^2 or Φ_1^2. A critical point is the states Φ_0^1 or $\Phi_0^2$are beliefs about the signaller being of one type, and states Φ_1^1, or $\Phi_1^2$are beliefs about the signaller being the other type. Types are as discussed earlier, high/ low, good/bad.

Once the states are updated agents can assign posterior probabilities to E happening as below:

For the first agent/ buyer in our case $q_{10}= I<\Phi_0^1, \psi>I^2 = 1/4IC_{00}+C_{01}I^2/(IC_{00}I^2+IC_{01}I^2)$, $q_{11}=1/4IC_{10}+C_{11}I^2/(IC_{10}I^2+IC_{11}I^2)$. Similarly, we can compute for the other agent, q_{20} and q_{21}. Hence posterior probabilities for each agent is an updation of belief about the types of the signaller. The numerator terms can be considered as to be interference terms which are responsible for the violation of Aumann's theorem, otherwise the posteriors of the two agents in this case would be same always.

Hence here we have two critical conditions, one, for the existence of the common knowledge as the unit operator, we have $q_{i0} =q_{i1}$, with i=1,2, however posterior probabilities are different across the agents. Hence following Haven, Khrennikov and Robinson (2017) to maintain the above conditions we arrive at the following sets of non-linear equations:

$C_{00}C_{01}/(C_{00}^2+C_{01}^2) = C_{10}C_{11}/(C_{10}^2+C_{11}^2)$ and,

$C_{00}C_{10}/(C_{00}^2+C_{10}^2) = C_{01}C_{11}/(C_{01}^2+C_{11}^2)$, Again for finding solutions we can equate ratios of coefficients to real no, for eg:$C_{00}/C_{01} = C_{10}/C_{11} = K$, where k >0 and is a real no.

3. Discussion

Remark 4: costly signalling fails generally under the Quantum-like regime

Buyers will be interested in buying from the seller if the belief state of the market is that the seller is of good type with a higher probability. Hence in our case, costly signalling can only be created when the market updates the belief statements about the reputation of the signaler, again reputation here means the probability of being a good type seller, whose success probability is higher than that of the bad type seller[15].

Hence the only way costly signalling may still work given that agents are quantum-like is if still posterior probabilities across agents are equal and common knowledge, i.e. Auman's theorem still holds given that agents update based on quantum rules.

As shown by Khrennikov (2015), the specific condition for the Auman's theorem not to hold is the incompatibility between information representations of agents and the events, or the prior states, i.e.

$[A^{(i)},E] \neq 0$ or $[A^{(i)}, \varrho] \neq 0$. However, it can also be shown that even if the non-commutation holds but the interference terms contributed by the individual agents cancel each other Auman's theorem is still valid. Hence even if agents are quantum-like cancellation of interference terms can still make costly signalling work, since market still will be able to agree on the updated probability of the seller's type, and trade may continue.

Remark 5: costly signalling can be recovered under special conditions/ nature of the common prior state

We have defined costly signalling as the phenomenon when there is a significant updating of belief states of the agents[16] (i.e. the probability of the seller being a good/ bad type) such that

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[15] *But here* $\sum q_{i0} = \sum q_{i1}$, may be a real solution, which in other words mean that the market is undecided between the types of the signaler or the seller.

Hence the above equality shows that the purpose of costly signalling is failed. No matter how much the posterior probability differs from each other or the prior, the market as a whole in this model in very special cases be indifferent between type 0 and 1 of the sellers!

[16] In our model the updating of belief state is about the reputation of the seller, in standard information asymmetry models adverse selection problem in the market prevents investors to distinguish between low type and high type firms/ sellers of products/financial products, due to which buyers/ investors misprice, which again may discourage the high type of sellers to remain in the market. Hence to solve such an unwanted outcome, or bad equilibrium, the seller of high quality sends a costly signal, which can not be easily copied by a low type.

the posterior probabilities are common knowledge and equal. However, our agents are Quantum-like. This can only happen if for the particular solutions for the system of equations () we choose k = 1, which again means choosing $\alpha = ½$, in appendix a detailed calculation is presented.

This choice automatically means ϱ=IΦ><ΦI, should have a pure state form only, more specifically in this model Φ = ½ [I00>+I01>+I10>+I11>].

More generally posteriors can also be equal and a common knowledge if the ‘interference terms’ for all agents cancel each other out (details in appendix).

Hence, we have two important results overall which can be contrasted with the Bayesian modelling outcomes:

There can be significant updating from a very low prior, and Costly signalling can occur only when we start from a maximally uncertain prior, or a pure state.

Remark 6, Scenario 3

Use of POVM

We consider here the scenario where the seller/ signaler in the model has incomplete knowledge about the type/ quality of the product, though this level of knowledge is at least higher than that of the buyers or the final decision-makers in the model. In such a scenario there can be ambiguity in signalling, or in other words, the signaler may not be fully sure of what type of signal is to be sent to convey the quality.

In such a scenario, we may assume the following set up.

Let’s us consider two different prior states[17], s_1 and s_2, and based on each prior state the signaler sends two different types of signals (events as in the model). The quantum-like agents update the posteriors based on the above rules, and as in the last case, there can be common posteriors in all these cases. However, since the signalling is itself ambiguous there is no certainty that the posterior beliefs would be updated favourably. The buyers / DM can either continue buying (action a_1) or stop buying (action a_2).

Hence, we can formulate the matrix of transition probabilities (Haven and Khrennikova, 2018):

$\begin{pmatrix} p(a1,s1) & p(a1,s2) \\ p(a2,s1) & p(a2,s2) \end{pmatrix}$ Where p(a,s) is the conditional probability of doing action a_i, when the prior state is s_i.

Aumann's theorem then helps to form a separating equilibrium wherein the equilibrium all agents agree with the updated belief about the quality of the seller/ asset in question.
Our model can be perceived as a general model where under special conditions Aumann’s theorem can be preserved given that the updating of beliefs happen in a non-Bayesian way.

[17] Here we assume that the prior states are as perceived by the signaler, this state is rather epistemological since complete information is not available, hence the ambiguity in signaling.

When the transition matrix is not doubly stochastic, i.e. $p(a_1,s_1)+p(a_2,s_1)$ and, or, $p(a_1,s_2)+p(a_2,s_2)$ is not unity, the basis vectors describing the decision makers belief state need not follow orthogonality condition. In such a case the belief updating process cannot be described by typical orthogonal projection measures. Rather more generalized positive semidefinite operators are required, often termed as POVM.

In the appendix, a brief introduction to POVM is provided. Recently (Haven and Khrennikova, 2018) have provided a formulation of expressing generalized POVM in terms of standard orthogonal basis, hence POVMs expressed in orthogonal basis can again be used to operate on the initial belief states of the decision makers to obtain the updates in belief states or probabilities.

Remark 7: *price implications*

Following the standard Bayesian modelling as in Isaac (2003), we can compute the seller's value from continuing to trade, based on the current period reputation and updating of the same based on signalling. Here too three scenarios can be conceived of, symmetric and no information at all about the seller's quality, asymmetric and full knowledge about the quality with the seller only, and seller being partially knowledgeable about the quality.

In the first scenario, with seller being ignorant of her type as the buyer is, and the trading being in a finite horizon model(which means in the first period itself it will be decided whether the trade will further continue or not) Standard Bayesian model (Issac, 2003) will allow the updating of the initial reputation λ, and so will the Quantum updating formula, however the expression for the formula for total probability will be different in both cases. Where the denominators in both the formulas is the total probability expression. However once quantum probability frame work is adapted the formula for total probability is revised, with additive perturbative terms (details in appendix),

Remark 8: *an explicit example of pricing formula*

Efficiency market theory in finance is a seminal theory of asset pricing based on deductive reasoning of homogenous rational agents, who perfectly anticipate future payoffs like dividends, and thus incorporate the same in present prices. Such exercise (though very theoretical) can be represented by the famous no arbitrage pricing formula, or the fair pricing formula, or more technically a Martingale. Where the price is simply the correctly anticipated present value of future pays (like dividends) at the correctly anticipated discount rate:

One general formulation is $p_t = \lambda \sum w_{j,t} (E_j(d_{t+1}|I_t)+ E_j(p_{t+1}|I_t))$[18], where p's are the prices at time t and expectations of prices at time t+1, d is the expectation of dividends paid at time t+1, E is the expectation operator, λ is the proper risk adjusted discount factor, and w's are the weights

[18] Arthur (2015) summarizes models in complexity finance which have used this formulation, but such models expand on the behavioral finance approach.

for the respective agent j, i.e. the relative confidence placed on jth agent's confidence of the dividends and the forward prices[19].

Hence to arrive at the above formulation the assumptions are: a. homogenous investors use the available information in the same manner or mechanism (for example the standard Bayesian mechanism) to form expectations of dividends, b. each agents knows that each agent knows….that each agent is rational, and thus arrive at the same expectation, i.e. rationality of agent is a common knowledge giving rise to homogenous expectation c. and that the price will be formed by the above equation is also a common knowledge.

Here certainly there can be sequence of such prices which are fair prices only based on random fluctuations in the information sequences.

Both behavioural finance experts and the complexity theory experts have challenged this formulation based on the assumption that agents (investors) are bounded rational and use inductive rather than deductive reasoning to arrive at any pricing value. Price dispersions, non-equilibrium prices, deviation of prices from fair value for a considerable period of time (Shiller, 1982) are some of the main implications of such heuristics and inductive reasoning-based models.

However, if we consider the above developed quantum like frame work, we can clearly find that generally even if the prior beliefs about expected dividends can be same, and through sequential updating of beliefs using the Born's principle rather than Bayesian formula, if the posterior beliefs about the expected dividends are a common knowledge, there can still be 'rational' disagreement about the same. Such rational disagreements can then generate non equilibrium prices, as also found in the behavioural models.

4. Conclusion

In the last decade, there has been an upsurge in interest about Quantum-like modelling in social sciences, there is already a growing body of empirical literature (as cited earlier) which provides data support to predictions of Quantum-like modelling in decision making. Khrennikov (2015) has pioneered the Quantum-like formulation of Auman's common knowledge theorem, which is actually a more general model since though generally Quantum-like agents can agree to disagree even if they start from a common prior and posterior probabilities are common knowledge, in special cases Auman's original theorem can still be recovered.

The current paper provides an early attempt of extending the Quantum-like the formulation of common knowledge theorem to the central theory of financial economics, i.e. costly signalling in an uncertain market scenario. Bar-Issac (2003) in a similar Bayesian signalling model has observed that in a monopoly market where the seller is perfectly informed about the quality of the product/ say asset in case of a corporate firm, there can be full information revelation via

[19] This formulation is used in many behavioral models, as well as models based on complexity theory, eg in Arthur (2015).

the costly signalling (which in the current model is the decision to continue trading). However, in the case of oligopolistic competition, partial information revelation is possible. We show here that if agents are Quantum-like rather than Bayesian signalling can be more cost effective, since, one, the zero prior trap can be avoided, and two, signalling solution (i.e. rational agents updating their beliefs about the reputation of the seller) can occur when the common prior is an uncertain state, which has no parallel in the standard Bayesian paradigm. Summarily we can draw the following three important points of departure from the standard decision-making modelling:

Generally, if agents/ DM are quantum-like or update belief states based on Quantum probability rules, then ‘zero prior' trap of Bayesian model can be avoided.

Under the special circumstance of starting from the pure uncertain prior state even Quantum-like agents can agree on posterior belief states which are common knowledge, hence updating the belief state for the signaller and making the costly signal to work.

When the signaller and the DMs are both partially informed about the quality types there can be ambiguity in signalling which cannot be described by standard orthogonal projector-based measures, POVM needs to be invoked. POVM can be used in the very set up based on other conditions also, for example (Yearsley, 2017) if the decision making exhibits order effects which is more evident under ambiguity environment, or when the responses from the decision makers (DM) can be more than the basis states as in the model, which can also be perceived in the state of ambiguity (Yearsley, 2017).

One of the major advantages of the quantum decision theory set up is that we don’t have to necessarily assume irrationality in decision making for explaining different disequilibrium behaviours of markets. Even if agents are perfectly rational the extended common knowledge set up demonstrate choice or decision outcomes which are not typically found in ‘efficiency market’ theory.

Contextuality studies are also deeply related to the violation or not violation of Aumann’s theorem, for example as demonstrated in the paper under specific contexts of compatibility Aumann’s theorem can be held valid, which would then generate equilibrium price behaviours in financial markets.

Last but now the least, QBism approach to the common knowledge problem can be quite intriguing, one interesting difference is that in the above discussed quantum approach the prior probabilities for any event, is still objective and known to every rational agent, however in the QBism approach probabilities are always personal degree of beliefs.

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5. APPENDIX

Appendix 1

General description of common knowledge operator in Quantum-like modelling is provided (as in Haven, Khrennikov and Robinson, 2018) analogically to the classical theory. In the classical theory set of all states of the world where an event E is a common knowledge is denoted by kE.

In quantum like modelling this definition is generalised, here kE is a projector on the subspace of all states of the world where E is a common knowledge. As in the set theory a system of projectors is defined,

Again, if kE is not 0, then it is an element of the system of projectors. The next step is to formulate the quantum version of belief or information update of agents, since in the classical or Bayesian framework this updating rule plays the central role in Auman's theorem.

In the classical set theoretic framework, rational agents agreeing on priors mean a very detailed probabilistic description of state of affairs, which specifically mean a collection of conjunction of probabilities is possible, of the form $p_{j1\ldots jN} = p(P^{I}_{ji} \wedge P^{2}_{j2} \wedge \ldots \wedge P^{N}_{jN})$ exists. However in the quantum like version such a description may not be possible since the information partitions of agents need not be compatible with each other, or in other words $P^{I}_{ji} \wedge P^{2}_{j2} \wedge \ldots \wedge P^{N}_{jN} = 0$ is a possibility. Again, this possibility or feature is based on the structure of quantum logic, mainly the absence of distributive property.

Hence as it is well noted in the literature that quantum prior provides much less information as compared to the classical prior, or in other words, information gain based on agreeing on quantum prior is less. Authors (Haven, Khrennikov, and Robinson, 2018) note that one may speculate that one of reasons for difference between quantum like version of the common knowledge theorem and the classical version is the above.

However, as demonstrated by Khrennikov (), Aumann's theorem can be violated by quantum like agents even if the information representations of agents are compatible. In the compatibility case the probabilities $p_{j1\ldots jN} = Tr\,(\rho\, P^{I}_{ji} \wedge P^{2}_{j2} \wedge \ldots \wedge P^{N}_{jN})$, where ρ is the density matrix representing the common world state.

Appendix 2

The revised formula for total probability in Quantum Frame work:

Modified formula for total probability, or law of total probability, LTP, has been used in many scenarios to explain human decision making under contexts like uncertainty (quite well summarised in 'quantum social science' by Haven and Khrennikov (2013)). Both order effects and interference terms in LTP can be demonstrated using POVM

Consider two generalized observables a and b corresponding to POVMs $M_a = \{V^*_j V_j\}$ and $M_b = \{W^*_j W_j\}$, where $V_j \equiv V(\alpha_j)$ and $W_j = W(\beta_j)$ correspond to the values α_j and β_j. If there is given the state ρ the probabilities of observations of values α_j and β_j have the form

$$p_a(\alpha) = Tr\rho M_a(\alpha) = TrV(\alpha)\rho V^*(\alpha),\ p(\beta) = Tr\rho M_b(\beta) = TrW(\beta)\rho W^*(\beta).$$

Now we consider two consequtive measurements: first the a-measurement and then the b-measurement. If in the first measurement the value a = α was obtained, then the initial state ρ was tranformed into the state

$$\rho_{a\,\alpha} = V(\alpha)\rho V^*(\alpha) / (TrV(\alpha)\rho V^*(\alpha))$$

For the consequtive b-measurement, the probability to obtain the value b = β is given by

$p(\beta|\alpha) = Tr\rho_a(\alpha)M_b(\beta) =$

$$TrW(\beta)V(\alpha)\rho V^*(\alpha)W^*(\beta)/(\ TrV(\alpha)\rho V^*(\alpha))$$

This is the conditional probability to obtain the result b = β under the condition of the result a = α. We set $p(\alpha,\beta) = p_a(\alpha)p(\beta|\alpha)$.

Now since operators need not commute $p(\alpha,\beta) = p(\beta, \alpha)$

We recall that, for two classical random variables a and b which can be represented in the Kolmogorov measure-theoretic approach, the formula of total probability (FTP) has the form $p_b(\beta) = \sum p_a(\alpha)p(\beta|\alpha)$.

Further we restrict our consideration to the case of dichotomous variables, $\alpha = \alpha_1,\alpha_2$ and $\beta = \beta_1,\beta_2$.

FTP with the interference term for in general non-pure states given by density operators and generalized quantum observables given by two (dichotomous) PVOMs:

$$p_b(\beta) = p_a(\alpha_1)p(\beta|\alpha_1) + p_a(\alpha_2)p(\beta|\alpha_2) + 2\lambda\sqrt{p_a(\alpha_1)p(\beta|\alpha_1)p_a(\alpha_2)p(\beta|\alpha_2)},$$

or by using ordered joint probabilities $p_b(\beta) = p(\alpha_1,\beta) + p(\alpha_2,\beta) + 2\lambda_\beta\sqrt{p(\alpha_1,\beta)p(\alpha_2,\beta)}$. Here the coefficient of interference λ has the form: $\lambda = Tr\rho\{W^*(\beta)V^*(\alpha_i)V(\alpha_i)W(\beta) - V^*(\alpha_i)W^*(\beta)W(\beta)V(\alpha_i)\} / \quad 2\sqrt{p_a(\alpha_1)p(\beta|\alpha_1)p_a(\alpha_2)p(\beta|\alpha_2)}$ Introduce the parameters

$$\gamma_{\alpha\beta} = Tr\rho W^*(\beta)V^*(\alpha)V(\alpha)W(\beta) / (Tr\rho V^*(\alpha)W^*(\beta)W(\beta)V(\alpha))$$

$$= p(\beta,\alpha) / p(\alpha,\beta)$$

This parameter is equal to the ratio of the ordered joint probabilities of the same outcome, but in the different order, namely, “b then a” or “a then b”. Then,

Interference term λ = ½ {√(p(α1, β)/p(α2,β) *(γα1β -1) + √(p(α2, β)/p(α1,β) *(γα2β -1)

In principle, this coefficient can be larger than one. Hence, it cannot be represented as λ = cosθ for some angle (“phase”) θ, cf.. However, if POVMs Ma and Mb are, in fact, spectral decompositions of Hermitian operators, then the coefficients of interference are always less than one, i.e., one can find phases θ.

Appendix 3

Positive operator value measures (POVM): recently POVM has been used in cognitive modelling related to describing choice behaviour of agents under uncertainty, this is a very helpful tool in describing agents' behaviour in case of uncertainty in financial markets since many interesting results like order effects can be explained. Authors () point out that positive operators are increasingly used to model decision making since in real life scenarios there can be noise in the decision-making process.

Positive operators are a class of projection operators which have more general properties, for example, if E is one positive operator then it can be conceived of as E = M'M, where M is a self-adjoint operator and M' is the transpose conjugate of M, such that for all such observations ∑M'M= I where I is the identity operator. Again, M can be given a square matrix representation, for example, if ϵ is the noise in the system then M = $\begin{bmatrix} \sqrt{1-\epsilon} & 0 \\ 0 & \sqrt{\epsilon} \end{bmatrix}$.

Noise in the system has an important interpretation in the decision theory literature; for example, say due to some noise in the final choice action, or due to some error, the agent rather choosing the optimal chooses a wrong option, now such actions can be represented by positive operators, rather than more stringent projection operators as described earlier.

There are several interesting properties of positive operators (Yearsley, 2017), such as: they are non-repeatable (E^2 is not equal to E), they are not unique, they are used when the basic elements in the Hilbert space of the model need not be orthogonal, they are used when there are more responses than there are basis states, this last property can be used in the decision making models with noise in the system.

Hence, A positive operator valued measure (POVM) is a family of positive operators {Mj} such that Pm j=1 ∑Mj = I, where I is the unit operator. It is convenient to use the following representation of POVMs: Mj = V* j Vj, where Vj: H → H are linear operators. A POVM can be considered as a random observable. Take any set of labels α1,...,αm, e.g., for m = 2,α1 = yes,α2 = no. Then the corresponding observable takes these values (for systems in the state ρ) with the probabilities p(αj) ≡ pρ(αj) = TrρMj = TrVjρV* j. We are also interested in the post-measurement states. Let the state ρ was given, a generalized observable was measured and the value αj was obtained. Then the output state after this measurement has the form

ρj =VjρV * j /(TrVjρV* j).

Hence, we see that the agents may still update the belief states following the same Born's rule.

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