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Measurement, Luders and von Neumann Projections and Nonlocality

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Abstract

The nature of the Lüders projection and its relationship with the von Neumann projection are clarified. It is shown that these two postulates apply under mutually exclusive and complementary conditions. It is then shown that Lüders transformations can also be obtained by unitary transformations for single and product states, but not entangled states. Several examples are given to illustrate this. The distinction between the two projections is further clarified by analyses of the violation of Einstein locality for both single particle and entangled states. A possible experimental test of the Lüders postulate is also proposed.

1 Introduction

Quantum mechanics has been plagued with the measurement problem (and nonlocality) almost since its inception. This basically stems from the conjunction of two hypotheses which are incompatible with each other. The first hypothesis is that quantum mechanics is a universal theory of everything, including the measuring apparatus. Consequently, the measurement interaction between a quantum mechanical system S and the apparatus A used to measure it inevitably produces an entangled state $|\Psi\rangle_{SA} = \sum_i c_i|\Psi_i\rangle_S \otimes |\Psi_i\rangle_A$, $\sum_i |c_i|^2 = 1$ in which the system and apparatus cease to have definite and independent states of their own. Nevertheless, definite measurement outcomes $|\Psi_i\rangle_S \otimes |\Psi_i\rangle_A$ are empirically observed, indicating abrupt state changes not explainable by Schrödinger evolution. To account for such changes, von Neumann introduced the second hypothesis, the operation of a non-quantum mechanical process which depends on the measured value which is unpredictable [1]. It is known as the 'projection postulate'. In the words of Schrödinger [2], "every measurement suspends the law governing the steady change in time of the ψ-function, and

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brings about an entirely different change, which is not governed by a law but by the result of the measurement." This is essentially the measurement problem. It is the ad hoc nature of the projection postulate and its clash with Schrodinger evolution that has spawned the plethora of interpretations of quantum mechanics [3].

To be more specific, let us consider the pure state $\rho = |\Psi\rangle_{SA}\langle\Psi|_{SA}$. It satisfies the conditions $\rho^2 = \rho$, $\mathrm{Tr}\rho = 1$. This contains off-diagonal interference terms in the presence of which it is not legitimate to conclude that the states $|A\rangle_j$ are definite eigenstates of the observable. Hence a second measurement has to be performed on the apparatus states, leading again to similar conclusions, requiring a third measurement, and this chain is obviously non-terminating. It has been argued that this infinite regress must eventually culminate in a 'non-quantum mechanical' subjective perception [1]. von Neumann represented this termination by an acausal process in the following way. If an observable $\hat{O} = \sum_i a_i P_i$ with discrete and nondegenerate eigenvalues a_i and projectors $P_i = |i\rangle\langle i|_S \otimes |i\rangle\langle i|_A$, is measured on the system, then according to the von Neumann projection postulate the state changes to

$$\rho \rightarrow \hat{\rho} = \sum_i P_i \rho P_i \tag{1}$$

The resulting state $\hat{\rho}$ is a mixed state, i.e. $\hat{\rho}^2 = \hat{\rho}$, $\mathrm{Tr}\hat{\rho} = 1$, showing that the projection process is non-unitary. This is an ideal measurement in quantum mechanics and works for complete and orthogonal basis vectors. Exactly at which stage the termination is to be applied is, however, left unspecified and arbitrary.

In 1951 Lüders [4] introduced a supplementary postulate which he claimed leads to the same mathematical consequences as the von Neumann projection postulate (1) in certain cases, and at the same time clarifies the nature of the abrupt state change on measurement or 'wave function reduction'. The Lüders postulate states that

$$\rho \rightarrow \rho_k^{\prime} = \frac{P_k \rho P_k}{\mathrm{Tr} P_k \rho} \tag{2}$$

on the condition that the result a_k was obtained. In the extreme case of $P_k = I$, the identity operator, this gives $\rho \rightarrow \rho$, which is what one would expect: there is no state change and a pure state remains a pure state. This is where the Lüders rule differs from a von Neumann projection which, even in the case of the identity operator, would result in a mixed state $\hat{\rho} = \sum_i |i\rangle\langle i|$ with the same weight for all states. The Lüders rule describes a change from a state ρ to a specific pure state $\rho_k^{\prime}$ in Hilbert space. The rule preserves unitarity and coherence and updates pure states to pure states. According to Lüders it 'shows exactly what is meant by the expression "reduction of the wave function".' This meaning will become gradually clear in the rest of the paper, particularly in Section 5.

However, if one considers the total state without selection or reading of individual results, the state transforms to

$$\hat{\rho} = \sum_k p_k \rho_k^{\prime} = \sum_k P_k \rho P_k \tag{3}$$

where $p_k = \mathrm{Tr} P_k \rho$ is the probability weight of the state $\rho_k^{\prime}$ in the full ensemble [5]. This is a statistical mixture of states identical with that given by the von Neumann rule (1), and clarifies its meaning and applicability.

It is important to emphasize that Lüders' and von Neumann's schemes for updating differ markedly for degenerate quantum systems. Lüders raised two concerns regarding the von Neumann ansatz. The first concern was that if an observable A is the fully degenerate identity, then it should be that $\rho' = \rho$. The second concern was that the transformation should depend only on the $\{P_k\}$. Together, these led him 'almost inevitably' to his ansatz. On the other hand, von Neumann's scheme for degenerate systems lies in 'lifting' the degeneracy. If an observable A has degenerate discrete eigenvalues, then first of all a refinement A' is found which commutes with A and has only non-degenerate discrete eigenvalues (such that $A = f(A')$, and eigenvalue a of A is also $f(a')$ where a' is an eigenvalue of A') and measurement is performed on it, lifting the degeneracy in the process. Hence, in von Neumann's ansatz the projection operators are typically one-dimensional projectors on the subspaces of the refinement observable, and all coherence is lost. As a passing remark here, we mention that such degenerate eigenvalue systems are normal in the 'cognitive' arena where Lüders' rule has been recently discussed.

However, it is not the case that the two methods differ only in the case of degenerate systems. The very experimental conditions under which the two postulates apply are mutually exclusive. The von Neumann postulate is applicable only when no selection is made or reading taken of individual measurement results. On the other hand, the Lüders rule is applicable only to selected eigenstates after the measurement, and therefore implies 'conditionalization'. To illustrate the mutual exclusiveness of the applicability of the two postulates, consider the two ways in which the double slit experiment can be done. One can either (a) send an entire ensemble of identical particles one-shot through the apparatus and record the interference pattern on the final detector screen, or (b) watch the pattern grow by accumulation of single dots (detections) on the screen over a very long exposure time. The von Neumann postulate is applicable to the former case and the Lüders postulate to the latter. These two postulates are therefore contextual and complementary. This understanding of the nature of Lüders' rule and the conditions under which it is applicable opens up the possibility of studies of post-selected states in quantum mechanics that were not possible before, and which we will consider later in Section 4.

More generalized measurements can be defined in quantum mechanics by POVMs (Positive Valued Operator Measures), a collection of positive operators $E_i \geq 0$ satisfying the conditions $\sum_i E_i = I$. Such a measurement is denoted by $M = \{E_i\}$. Each E_i is associated with an outcome of the measurement and since $E_i \geq 0$, it has the decomposition $E_i = M_i^\dagger M_i$. For a state ρ (pure or mixed), the probability of obtaining the result associated with E_i is $\mathrm{Tr}(E_i\rho)$. The post measurement state after obtaining the result i is

$$\rho \rightarrow \frac{M_i \rho M_i^\dagger}{\mathrm{Tr}(E_i\rho)}. \tag{4}$$

The main difference between these projections and projections like P_i which are orthogonal ($P_i P_j = P_i \delta_{ij}$) is that the M_i or E_i are not orthogonal. However, by Neumark's dilation theorem a POVM can be obtained from projective measurements acting in a larger Hilbert space [6].

2 The Mathematical Nature of Lüders Transformations

(i) Finite dimensional Hilbert spaces

In order to illustrate the mathematical nature of Lüders transformations in a finite dimensional Hilbert space, let us consider the state

$$|\psi\rangle = \sum_i c_i|\psi\rangle_i \tag{5}$$

where $|\psi\rangle_i$ form a complete set of orthonormal eigenvectors of orthogonal projectors $P_i = |\psi\rangle_i\langle\psi|$, and $\sum_i |c_i|^2 = 1$ so that $|||\psi\rangle|| = 1$. Then,

$$P_i|\psi\rangle = c_i|\psi\rangle_i, \quad ||P_i|\psi\rangle|| = |c_i|^2 < 1 \tag{6}$$

Thus, the norm is not preserved on projections. Furthermore, a projector has no inverse. Consider now a unitary transformation

$$U|\psi\rangle = |\psi\rangle' = \sum_i c_i U|\psi\rangle_i, \quad ||U|\psi\rangle|| = \sum_i |c_i|^2 = 1. \tag{7}$$

The norm is preserved in this case.

Now consider the Lüders transformation

$$|\psi\rangle'_i = \frac{P_i|\psi\rangle}{c_i} = |\psi\rangle_i \tag{8}$$

so that

$$|||\psi\rangle'_i|| = |||\psi\rangle_i|| = 1. \tag{9}$$

Here the norm is preserved. However, defining $P'_i = P_i/c_i$, we get $P'^2_i = P_i^2/c_i^2 = P_i/c_i^2 \neq P'_i$. Hence, strictly speaking, P'_i is not a projector, but nevertheless, involving P_i as it does, it has no inverse.

However, it is always possible to define a state dependent unitary transformation

$$U(\varphi_i)|\psi\rangle = |\psi\rangle_i, \quad |||\psi\rangle_i|| = 1 \tag{10}$$

where φ_i are $d - 1$ angular coordinates of the state $|\psi\rangle_i$ which gives the same result as the Lüders transformation, except in cases where $|\psi\rangle$ is entangled because such states are basis independent.

We will now illustrate this through a number of concrete examples.

(ii) A rotation in R^2

To understand what is involved in (10), it will help to start with the simple case of the projections of a vector in R^2,

$$|\psi\rangle = \cos\theta|i\rangle + \sin\theta|j\rangle = \cos\theta\begin{pmatrix}1\\0\end{pmatrix} + \sin\theta\begin{pmatrix}0\\1\end{pmatrix} := a_i|i\rangle + a_j|j\rangle \tag{11}$$

where $|i\rangle, |j\rangle$ are unit vectors in the x and y directions. Let $A = a_i P_i + a_j P_j$ $(a_i = a_j)$ where $P_i = |i\rangle\langle i|, P_j = |j\rangle\langle j|$ are projectors with the properties $P_i^2 = P_i, P_j^2 = P_j, P_i P_j = 0, P_i + P_j = I$. Then,

$$P_i|\psi\rangle = a_i|i\rangle, \quad P_j|\psi\rangle = a_j|j\rangle. \tag{12}$$

Let us now define the vectors

$$|\psi\rangle'_i = \frac{P_i|\psi\rangle}{a_i} = |i\rangle, \quad |\psi\rangle'_j = \frac{P_j|\psi\rangle}{a_j} = |j\rangle. \tag{13}$$

These transformations are norm preserving but, as we saw above, they have no inverses and are therefore not unitary. However, they can clearly be replaced by rotations of the vector by the angle θ and by the angle $(\theta + 3\pi/2)$ in the clockwise direction, the rotation matrix being

$$\begin{aligned} \hat{U}(\theta) &= \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix} \\ \hat{U}(\theta)|\psi\rangle &= |i\rangle, \quad \hat{U}(\theta + 3\pi/2)|\psi\rangle = |j\rangle. \end{aligned} \tag{14}$$

Clearly, in each case the vector 'rotates' so as to align itself 'fully' with one of the basis vectors $|i\rangle$ and $|j\rangle$, there being no component along the other.

Let us now define $\rho = |\psi\rangle\langle\psi|$ which has the property $\rho^2 = \rho$. Then

$$\rho_i^{\psi} = \frac{P_i \rho P_i}{a_i^2} = |i\rangle\langle i|, \tag{15}$$

$$\rho_j^{\prime} = \frac{P_j \rho P_j}{a_j^2} = |j\rangle\langle j|. \tag{16}$$

These are Lüders updates in R^2. One can now define an operator

$$\hat{\rho} = a_i^2 \frac{P_i \rho P_i}{a_i^2} + a_j^2 \frac{P_j \rho P_j}{a_j^2} = P_i \rho P_i + P_j \rho P_j. \tag{17}$$

This is the von Neumann projection.

Consider now the vector ψ with $a_i = a_j = a = 1/\sqrt{2}$ (degenerate case) and $\hat{O} = (P_i + P_j) = I$. An identity operator cannot change a vector, and according to the Lüders ansatz, $|\psi\rangle \to |\psi\rangle$ and $\rho \to \rho$. But the von Neumann ansatz implies

$$\hat{\rho} = a^2(|i\rangle\langle i| + |j\rangle\langle j|) = \frac{1}{2} I. \tag{18}$$

This is the reason von Neumann had to think of of 'lifting' the degeneracy whenever it occurred before applying his rule.

(iii) Single qubit states

Let us now consider a single qubit state. It can be written as a vector

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \tag{19}$$

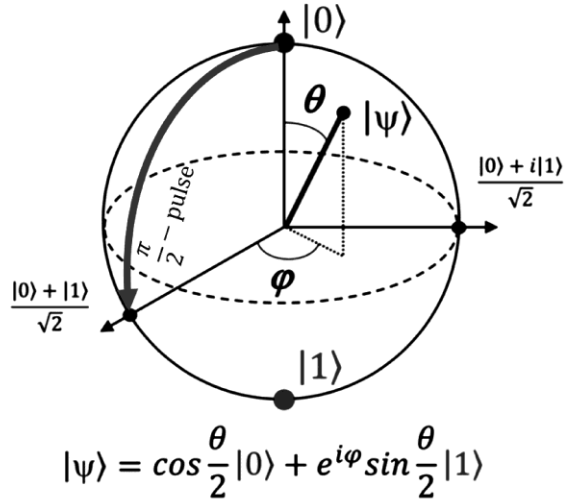

Figure 1: The Bloch Sphere

of the Bloch sphere. Suppose one measures $\sigma_z = |0ih0| - |1ih1| := P_0 - P_1$ on the state. There are two possible results, + or −, with the probabilities

$$\mathbf{Pr}\{+1\} = \mathrm{Tr}P_0\rho = |h\psi||0ih0||\psi i|^2 = \cos^2\frac{\theta}{2}, \tag{20}$$

$$\mathbf{Pr}\{-1\} = \mathrm{Tr}P_1\rho = |h\psi||1ih1||\psi i|^2 = \sin^2\frac{\theta}{2}, \tag{21}$$

and the post measurement selected states are, according to Lüders' rule,

$$\rho_0 = \frac{|0ih0||\psi ih\psi||0ih0|}{\mathrm{Tr}P_0\rho} = |0ih0|, \tag{22}$$

$$\rho_1 = \frac{|1ih1||\psi ih\psi||1ih1|}{\mathrm{Tr}P_1\rho} = |1ih1|. \tag{23}$$

As one can readily see from Fig. 1, the transformation in the first case can also be obtained by an anticlockwise rotation by angle $(\pi/2-\varphi)$ about the Z axis followed by an anticlockwise rotation by angle θ about the X axis, and in the second case by an anticlockwise rotation by angle $(2\pi - \varphi)$ about the Z axis followed by an anticlockwise rotation by angle $(\pi - \theta)$ about the Y axis. In each of these cases the original state with components along both basis states is rotated and aligned completely with one of the basis states.

In general an arbitrary rotation on the Bloch sphere can be written as the unitary operator

$$U(\theta,\varphi) = e^{i\varphi}\exp\left(-i\frac{\theta}{2}\boldsymbol{\sigma}.\mathbf{n}\right) \tag{24}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli spin vector and $\mathbf{n}$ is an arbitrary direction of the Bloch sphere [7, 8]. Hence, the normalized states produced by the non-unitary Lüders' rule applied to a qubit can also be obtained by rotations on the Bloch sphere.

When no post-selection is done, one obtains the mixed state

$$\hat{\rho} = \mathrm{Tr}(P_0\rho)\rho_0 + \mathrm{Tr}(P_1\rho)\rho_1 = \cos^2\frac{\theta}{2}\rho_0 + \sin^2\frac{\theta}{2}\rho_1 \tag{25}$$

Notice that classical probability theory cannot be applied to the pre-measurement pure state $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is given by (19), because it has off-diagonal elements which are interference terms. One can, however, generalize the definition of probability to take account of these interference effects by defining a new formula for total probability (FTP). Consider the state (19) corresponding to the point (θ, φ) on the Bloch sphere and another state corresponding to the point (θ, φ') and form their inner product. One then gets the FTP

$$FTP = p_0 + p_1 + 2\sqrt{p_1 p_2}\cos\delta \tag{26}$$

where $p_0 = \cos^2\theta/2$, $p_1 = \sin^2\theta/2$ and $\delta = \varphi - \varphi'$. Since $-1 < \cos\delta < +1$, the total probability can be both larger than and less than the classical probability $p_0 + p_1$.

(iv) Two qubit entangled states

Next, let us consider the 2-qubit Bell state

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B] \tag{27}$$

in the tensor product Hilbert space $H_A \otimes H_B$ of two systems A and B whose basis states are $|0\rangle_A|0\rangle_B, |1\rangle_A|1\rangle_B, |0\rangle_A|1\rangle_B, |1\rangle_A|0\rangle_B$. Suppose we want to measure the state of the system. Then there are two possible results, $|0\rangle_A|0\rangle_B$ or $|1\rangle_A|1\rangle_B$. The projection operators are $P_0 = (|0\rangle_A\langle 0|) \otimes (|0\rangle_B\langle 0|)$ and $P_1 = (|1\rangle_A\langle 1|) \otimes (|1\rangle_B\langle 1|)$. Then, the probabilities for the two possible post measurement states are

$$\mathrm{Tr}P_0\rho = |\langle\Psi^+|P_0|\Psi^+\rangle|^2 = \frac{1}{2}, \tag{28}$$

$$\mathrm{Tr}P_1\rho = |\langle\Psi^+|P_1|\Psi^+\rangle|^2 = \frac{1}{2}, \tag{29}$$

and the post measurement Lüders updates are

$$\frac{|0\rangle_A\langle 0|\otimes|0\rangle_B\langle 0||\Psi^\pm\rangle\langle\Psi^\pm||0\rangle_A\langle 0|\otimes|0\rangle_B\langle 0|}{\mathrm{Tr}P_0\rho} = (|0\rangle_A\langle 0|) \otimes (|0\rangle_B\langle 0|) := \rho_0, \tag{30}$$

$$\frac{|1\rangle_A\langle 1|\otimes|1\rangle_B\langle 0||\Psi^+\rangle\langle\Psi^+||1\rangle_A\langle 1|\otimes|1\rangle_B\langle 0|}{\mathrm{Tr}P_1\rho} = (|1\rangle_A\langle 1|) \otimes (|1\rangle_B\langle 1|) := \rho_1. \tag{31}$$

However, unlike in the previous examples, these updates cannot be obtained by unitary transformations because an entangled state is basis independent.

With no selection or reading of the measured states, one obtains the von Neumann mixed state

$$\hat{\rho} = [P_0\rho P_0 + P_1\rho P_1] = \frac{1}{2}[\rho_0 + \rho_1]. \tag{32}$$

Check that $\hat{\rho}^2 = \frac{1}{4}(\rho_0 + \rho_1) = \hat{\rho}$, which means the transformation is non-unitary.

Nate that the normalization of the Lüders updates is crucial in ensuring norm preservation, but the processes are not unitary.

3 Measurement: An Example to Clarify the Distinction Between von Neumann and Lüders Projections

Let us take the case of an observation designed to measure some observable Q of a particle with wave function $\psi_1(\vec{x}, t)$. Let the wave function of the apparatus be $\psi_2(y, t)$ where y is the relevant coordinate of the apparatus. The interaction Hamiltonian is taken to be

$$H_I = -gQp_y \tag{33}$$

where g is a suitable coupling parameter and p_y is the momentum corresponding to y. During the impulsive interaction the free evolution of the two systems can be ignored, and

$$i\hbar\frac{\partial\Psi}{\partial t} = -gQp_y\Psi = \frac{ig}{\hbar} Q\frac{\partial\Psi}{\partial y} \tag{34}$$

is a good approximation, and where, under the assumption of discrete eigenfunctions,

$$\Psi(\vec{x}, y, t) = \sum_q \psi_{1q}(\vec{x})\psi_{2q}(y, t), \tag{35}$$

$$Q\psi_{1q}(\vec{x}) = q\psi_{1q}(\vec{x}). \tag{36}$$

Initially the electron and the apparatus are independent, and hence

$$\Psi(\vec{x}, y) = \psi_{10}(\vec{x})\psi_{20}(y) = \psi_{20}(y)\sum_q c_q\psi_{1q}(\vec{x}). \tag{37}$$

The final wave function is given by

$$\Psi(\vec{x}, y, t) = \sum_q c_q\psi_{1q}(\vec{x})\psi_{20}(y - gqt/\hbar^2). \tag{38}$$

which is an entangled wave function embodying a correlation between the eigenvalue q of Q and the apparatus coordinate y.

A specific example would be spin measurement with $Q = \sigma_z$. In this case the apparatus is a dipole magnetic field in the xz plane with a gradient in the z-direction, and let

$$\psi_1(x, z) = (c_+|u_+\rangle + c_-|u_-\rangle) f(z)e^{ikx}, \tag{39}$$

$$\sigma_z|u_\pm\rangle = \pm|u_\pm\rangle \tag{40}$$

be the wave function of a neutral spin-$\frac{1}{2}$ particle travelling in the x-direction and passing through the magnet at $t = 0$, $f(z)$ being a Gaussian wave packet and $c_\pm$ the probability amplitudes for the spin-up and spin-down states $|u_+\rangle$ and $|u_-\rangle$ satisfying the condition $|c_+|^2 + |c_-|^2 = 1$. Then the initial wave function is

$$\Psi(x, z) = \psi_{20}(z)\psi_1(x, z) \tag{41}$$

and the interaction Hamiltonian is

$$H_I = i\hbar\mu_B \frac{\partial B}{\partial z}. \tag{42}$$

As a result of the impulsive action of the magnetic field, two wave packets are created which begin to separate in the z direction, and after sufficient time t the wave function has the form

$$\Psi(x, z, t) = [c_+ f_+(x, z, t)|u_+\rangle\psi_{20}(z_+) + c_- f_-(x, z, t)|u_-\rangle\psi_{20}(z_-)]\,, \tag{43}$$

where $f_\pm(x, z, t)$ are the evolved wave packets whose overlap is negligible and $z_\pm$ are the corresponding shifted coordinates of the apparatus [9]. This is a path-spin entangled state. In the standard theory of the Stern-Gerlach experiment, the particle enters a detector at $(x, y, z_\pm)$ and is absorbed, causing an irreversible macroscopic change in it, and these 'marks' are finally read out. The marks corresponding to the $z_\pm$ positions are then interpreted as measurements of the spin-up and spin-down states of the particle according to the spin-position correlations in the wave function (43).

Let us now see how this experiment is to be interpreted according to the Lüders postulate. Consider the spin projection operators $P_+ = |u_+\rangle\langle u_+|, P_- = |u_-\rangle\langle u_-|$. Then the Lüders updated pure states are

$$\frac{P_+\Psi(x,z,t)}{|c_+|} = f_+(x, z, t)|u_+\rangle\psi_{20}(z_+) := \Psi^+(x, z_+, t), \tag{44}$$

$$\frac{P_-\Psi(x,z,t)}{|c_-|} = f_-(x, z, t)|u_-\rangle\psi_{20}(z_-) := \Psi^-(x, z_-, t). \tag{45}$$

This shows that the system enters only one of the two possible channels at a time: $\Psi(x, z, t) \rightarrow \Psi^+(x, z_+, t)$ or $\Psi(x, z, t) \rightarrow \Psi^-(x, z_-, t)$, there being nothing in the other channel. However, the entangled state $\Psi(x, z, t)$ (43) being basis independent, there is no unitary transformation that can produce these updates. Contrast this with the case of the von Neumann projection according to which the wave function in one of the two channels vanishes abruptly and acausally on measurement. There is no such abrupt vanishing of wave functions on Lüders transformations which are norm preserving. The measurement problem is thus shifted from the problem of the dynamics of von Neumann projections to that of Lüders transformations neither of which the theory accounts for.

This opens up some new possibilities not hitherto explored, to the best of our knowledge. One can obviously add the Lüders updates in the two channels to obtain the pure state $\Psi^+(x, z_+, t) + \Psi^-(x, z_-, t)$ and combine them later to find out whether they interfere. We will explore this possibility in the next section to propose a test of the Lüders postulate.

4 Interference of Lüders Updates

Let us consider a photon in the polarization state

$$|\psi\rangle_i = \alpha|0\rangle + \beta|1\rangle \tag{46}$$

where $|0\rangle, |1\rangle$ are polarization basis states, and $|\alpha|^2 + |\beta|^2 = 1$. Let it be incident on a lossless 50 : 50 beam splitter. Then after the beam splitter its state can be written as

$$|\psi\rangle' = |\psi\rangle_i \frac{1}{\sqrt{2}}[i|a\rangle + |b\rangle] \tag{47}$$

where a denotes the reflected path and b the transmitted path. Let polarization detectors (D_a, D_b) be placed in the paths (a, b). Then

$$
\begin{aligned}
|\psi\rangle_f &= \frac{1}{\sqrt{2}}\left[i|a\rangle|\psi\rangle_i|D\rangle_a + |b\rangle|\psi\rangle_i|D\rangle_b\right] \\
&= \frac{1}{\sqrt{2}}\left[i|a\rangle(\alpha\,|0\rangle|D\rangle_{a0} + \beta\,|1\rangle|D\rangle_{a1}) + |b\rangle(\alpha\,|0\rangle|D\rangle_{b0} + \beta\,|1\rangle|D\rangle_{b1})\right] \\
&:= \frac{1}{\sqrt{2}}\left[i\alpha'\,|\psi\rangle_{a0}|D\rangle_{a0} + i\beta'\,|\psi\rangle_{a1}|D\rangle_{a1} + \alpha'\,|\psi\rangle_{b0}|D\rangle_{b0} + \beta'\,|\psi\rangle_{b1}|D\rangle_{b1}\right].
\end{aligned}
\tag{48}
$$

According to the Lüders rule, the states will be updated post measurement to the pure states

$$
\begin{aligned}
\frac{P_{a0}\rho_f P_{a0}}{|\alpha'|^2/2} &:= \rho'_{a0}\rho^D_{a0} \\
\frac{P_{a1}\rho_f P_{a1}}{|\beta'|^2/2} &:= \rho'_{a1}\rho^D_{a1} \\
\frac{P_{b0}\rho_f P_{b0}}{|\alpha'|^2/2} &:= \rho'_{b0}\rho^D_{b0} \\
\frac{P_{b1}\rho_f P_{b1}}{|\beta'|^2/2} &:= \rho'_{b1}\rho^D_{b1}
\end{aligned}
\tag{49}
$$

where $P_{\lambda i} = |\psi\rangle_{\lambda i}\langle\psi||D\rangle_{\lambda i}\langle D|$, $\lambda = (a, b), i = (0, 1)$. On completion of reading of the detector states, one obtains the complete ensemble of pure photon states

$$\rho' = \rho'_{a0} + \rho'_{b0} + \rho'_{a1} + \rho'_{b1}. \tag{50}$$

Let a post-selection be made of the states (ρ'_{a0}, ρ'_{b0}) or (ρ'_{a1}, ρ'_{b1}) and let them be combined on a second lossless 50 : 50 beam splitter. In terms of the updated state vectors, the states after the second beam splitter (omitting the primes henceforth) will be

$$
\begin{aligned}
|\psi\rangle^{out}_{ai} &= \frac{1}{2}\left[|\psi\rangle_{ai} + e^{i\theta}|\psi\rangle_{(b\to a)i}\right] = \frac{1}{2}[|\psi\rangle_{ai}(1 + e^{i\theta}), \\
|\psi\rangle^{out}_{bi} &= \frac{i}{2}\left[e^{i\theta}|\psi\rangle_{bi} - |\psi\rangle_{(a\to b)i}\right] = \frac{i}{2}[|\psi\rangle_{bi}(e^{i\theta} - 1)
\end{aligned}
\tag{51}
$$

where θ is an arbitrary phase that can be introduced by placing a phase shifter in arm b of the interferometer. Hence, $||\,|\psi\rangle^{out}_{\lambda i}||^2$ will contain the interference terms $\frac{1}{2}||\,|\psi\rangle_{\lambda i}||^2(1 \pm \cos\theta)$.

If one thinks of the classic double slit experiment with (a, b) denoting the paths, one cannot directly use the ket vectors because $|\psi\rangle_{ai}$ and $|\psi\rangle_{bi}$ are orthogonal projections and it would appear that there is no interference term. This is, of course, not true, and we will see in the last section how to deal with such cases algebraically. For the time being, consider the position vectors $|x\rangle$ and the projections $\langle x|\psi\rangle = \psi(x)$ where x stands for (x, y, z). Then, the wave function in the overlap region after the double slit can be written as

$$\psi(x)_i = \frac{1}{\sqrt{2}}\psi(x)_{ai} + e^{i\theta}\psi(x)_{bi} \tag{52}$$

where the overlapping wave functions $\psi(x)_{ai}$ and $\psi(x)_{bi}$ ($\psi(x)_{ai} \cap \psi(x)_{bi} = 0$) are not orthogonal, and hence

$$|\psi(x)_i|^2 = \frac{1}{2}\left[|\psi(x)_{ai}|^2 + |\psi(x)_{bi}|^2 + e^{i\theta}\psi^*(x)_{ai}\psi(x)_{bi} + e^{-i\theta}\psi^*(x)_{bi}\psi_{ai}\right], \tag{53}$$

which can be written in terms of quantum probabilities as

$$P_i = P_{ai} + P_{bi} + 2\sqrt{P_{ai}P_{bi}}\cos\theta \tag{54}$$

This the formula for total probability (FTP) in quantum mechanics for the case of the double slit. The interference term is a perturbation on the classical probability $P_{ai} + P_{bi}$ [10, 11]. This is the most natural and consistent interpretation of the quantum double slit experiment.

One may wonder why polarization-path entangled states are necessary to observe the interference of updated states. The reason is the following. The state $|\psi\rangle'$ (47) is in a tensor product Hilbert space $H_P \otimes H_p$ where H_P is the 2-dimensional complex Hilbert space of polarization states and H_p is the Hilbert space of paths. A projection in H_P cannot affect the phase of a photon because the phase is in H_p, being associated with path differences. It is well known that a path detection by any means wipes out the interference pattern, although there seems to be no general agreement about the mechanism of coherence loss.

A similar experiment can also be done with single neutrons where the basis states $|0\rangle, |1\rangle$ denote spin-up and spin-down states [12].

The absence of interference in such experiments will falsify the Lüders postulate.

5 Nonlocality

Finally, let us consider the implications of the Lüders update postulate regarding nonlocality. Quantum nonlocality has remained one of the biggest enigmas. Some authors maintain that there are two types of nonlocality in quantum physics, namely the Einstein type (highlighted in the 1935 EPR paper [13]) and Bell type [14]. It can be argued that while the first type of nonlocality is truly quantum in the sense of spooky action-at-a-distance, the latter is 'sub-quantum'. There have also been contextuality based interpretations of violations of Bell's inequality [18]. Most importantly, since the norm preserving Lüders updates do not imply 'wave function reduction' in the same sense as von Neumann projections, as we have seen, it seems fundamentally important to enquire what implication they have on nonlocality.

The thorny problem of nonlocality arose from the ontic interpretation of the quantum state. Traditionally one can think of two ends of the spectrum, statistical interpretation of the quantum state mainly due to Einstein, and the ontic interpretation mainly due to von Neumann. If one accepts the ontic interpretation as a whole, i.e., the quantum state is the physical state of a concrete quantum system, and the von Neumann projections are also physical, then nonlocality is inevitable, at least theoretically. For example, take the maximally entangled state of two qubits,

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B] \tag{55}$$

where A is with Alice and B with Bob who are spatially separated and at rest relative to each other, and $|0\rangle_A, |1\rangle_A$ and $|0\rangle_B, |1\rangle_B$ are the eigenvectors $\in H_A$ and H_B. At the time t of observation these qubits are two partial systems S_1 and S_2 which are "spatially separated and (in the sense of the classical physics) are without significant reciprocity" [15]. Now, according to Einstein, "on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is independent of what is done with the system S_1." This is Einstein's criterion of separability. However, according to the von Neumann projection postulate, if Alice measures her state at time t, she will get one of two possible results randomly, namely either

$$\frac{1}{\sqrt{2}}|0\rangle_A|0\rangle_B, \tag{56}$$

in which case the state $|1\rangle_A|1\rangle_B$ vanishes instantaneously, or

$$\frac{1}{\sqrt{2}}|1\rangle_A|1\rangle_B, \tag{57}$$

in which case the state $|0\rangle_A|0\rangle_B$ vanishes instantaneously. In both cases the vanishing of a part of the wave function changes the norm of the state. The mixed density matrix is given by

$$\hat{\rho} = \frac{1}{2}[\rho_0 + \rho_1]. \tag{58}$$

Both cases imply an instantaneous factual change of Bob's state, and hence a violation of Einstein's separability criterion. This has come to be known as nonlocality, a spooky action-at-a-distance or telepathy. The words spooky and telepathy used by Einstein are important here in emphasizing that this is different from the familiar Coulomb or Newtonian action-at-a-distance where there is a potential. There is no 'potential' here because, by hypothesis, the partial systems S_1 and S_2 are non-interacting at the time of the measurement [16]. Khrennikov [19] has claimed that such nonlocality can be resolved if probability or state updations are considered not to be happening in physical space.

Let us now examine the case from the point of view of the norm preserving Lüders updates which, as we have seen, are conditional updates and have a direct reference to the agent making the measurement. According to Alice the Lüders updated states are

$$|\Psi^+\rangle'_{0A} = \frac{P_0^A|\Psi^+\rangle}{1/\sqrt{2}} = |0\rangle_A|0\rangle_B, \tag{59}$$

$$|\Psi^+\rangle'_{1A} = \frac{P_1^A|\Psi^+\rangle}{1/\sqrt{2}} = |1\rangle_A|1\rangle_B. \tag{60}$$

In terms of the density matrix, one gets

$$\rho^{+\prime}_{0A} = |\Psi^+\rangle'_{0A}\langle\Psi^+|' = |0\rangle_A\langle 0||0\rangle_B\langle 0| = \rho_0, \tag{61}$$

$$\rho^{+\prime}_{1A} = |\Psi^+\rangle'_{1A}\langle\Psi^+|' = |1\rangle_A\langle 1||1\rangle_B\langle 1| = \rho_1, \tag{62}$$

it is clear from these that the state $|\Psi^+\rangle$ is transformed to the state $|0\rangle_A|0\rangle_B$ if Alice observes the state $|0\rangle_A$, or to the state $|1\rangle_A|1\rangle_B$ if she observes the state $|1\rangle_A$. This is a feature that is missing from the von Neumann projection (58). It is precisely this feature that clarifies the true meaning of 'wave function reduction': the 'conditional' norm preserving transformation $|\Psi^+\rangle \rightarrow |\Psi^+\rangle'_{0A}$ or $|\Psi^+\rangle \rightarrow |\Psi^+\rangle'_{1A}$. Hence, clearly there is no nonlocality of the type associated with von Neumann projection, even for ontic states.

Nevertheless, Alice's updates (59) and (60) indicate that Bob's states also get updated simultaneously. Since the state $|\Psi^+\rangle$ (55) is symmetric in (A, B), Bob's updates must also simultaneously update Alice's states, and so consistency requires that they be the same. Hence, there is spooky action-at-a-distance in the theory irrespective of the which projection postulate is adopted.

5.1 Single Particle Nonlocality

The origin of the debate on nonlocality in quantum mechanics can be traced back to Einstein's observations at the 1927 Solvay Conference, seven years before the 1935 EPR paper. Unlike the EPR paper, it deals with a single particle. Consider a single particle wave function suggested by him to demonstrate that an ontic wave function ψ for the particle and locality are incompatible [20]. After passing through a small hole in a screen, the wave function of the particle spreads out on the other side of it in the form of a spherical wave, and is finally detected by a large hemispherical detector. The wave function propagating towards the detector does not show any privileged direction. Einstein observes:

> If $|\psi|^2$ were simply regarded as the probability that at a certain point a given particle is found at a given time, it could happen that the same elementary process produces an action in two or several places on the screen. But the interpretation, according to which the $|\psi|^2$ expresses the probability that this particle is found at a given point, assumes an entirely peculiar mechanism of action at a distance, which prevents the wave continuously distributed in space from producing an action in two places on the screen.

Einstein later remarks that this 'entirely peculiar mechanism of action at a distance' is in contradiction with the postulate of relativity.

Let the stationary wave function of the particle be

$$\psi(R) = A\frac{e^{-ikR}}{R} \tag{63}$$

where R is the radius (suitably large) of the hemispherical detector and A is a normalization constant. Let $\psi(R) = \langle R|\psi\rangle$. Then the density matrix is $\rho = |\psi\rangle\langle\psi|$. The spherical symmetry makes the system degenerate. To 'lift' this degeneracy, let us for simplicity (but without loss of generality) represent the detector by a very large number N of discrete points (labelled by the polar and azimuthal angles (θ_i, φ_i)), and write the normalized state vector at each point as $|\psi\rangle_i$ with $\langle\psi|_j|\psi\rangle_i = \delta_{ij}$ and $|\psi\rangle = \sum_{i=1}^{N} c|\psi\rangle_i|D\rangle_i$ with $|c|^2 = 1/N$ because of spherical symmetry. Then, $\rho = |\psi\rangle\langle\psi|$. Tracing over the detector states, one obtains the von Neumann mixed density matrix

$$\hat{\rho} = \frac{1}{N}\sum_i |\psi\rangle_i\langle\psi| = \frac{1}{N}\sum_i \rho_i \tag{64}$$

This is a statistical mixture that describes a uniform probability distribution of particle detections over the hemisphere. Although the wave is spherical, the von Neumann projection prevents a particle from being detected in two places by collapsing the wave function everywhere except at the place of detection. This is a peculiar mechanism of action-at-a-distance.

To see what the Lüders postulate implies, one needs to send individual particles through the apparatus one at a time and record their detections. In that case, as pointed out earlier, the von Neumann projection is no longer applicable and one has to use norm preserving and conditional Lüders updates

$$\rho \to \rho_k = |\psi\rangle_k\langle\psi|. \tag{65}$$

Here the wave function does not 'vanish' anywhere–it is simply transformed into a single eigenfunction of position. This does not prevent action-at-a-distance, it only clarifies the nature of wave function collapse.

This type of nonlocality can be avoided only in epistemic interpretations of the wave function.

6 Concluding Remarks

We have shown why the von Neumann and Lüders projection postulates apply in mutually exclusive conditions, and are therefore contextual and complementary, and in the process clarified their precise nature. We have also shown through some explicit examples that the Lüders transformations can be replaced by equivalent unitary transformations in certain cases. However, this does not work for entangled systems and hence to measurement processes for which the Lüders transformation was suggested in the first place. Nevertheless it leads to an important clarification of the nature of wave function reduction/collapse, namely that it is a 'conditional' transformation of the initial pure state $\rho = |\Psi\rangle\langle\Psi|, |\Psi\rangle = \sum_i c_i|\psi\rangle_i$ to an individual post-selected 'normalized' pure state $\rho_k = |\psi\rangle_k\langle\psi|$ [21].

The distinction between von Neumann and Lüders projections has also been further clarified by analyses of the violation of Einstein locality in the case of both single particle and entangled states. Such nonlocality can only be removed by adopting some form of epistemic interpretation of the wave function. There exists a voluminous literature on this subject in which discussions on measurement and locality issues have involved various shades of ontic-epistemic interpretations of the quantum state [22].

Also, in discussions on locality physicists have adopted different definitions of locality which might or might not be fully equivalent to each other, as for example; 'no superluminal casual influences', or 'no spooky action between space like separated events', or 'no superluminal signal propagation', or 'space-like separated local observables commute'. Even Bell had two separate theorems. The 1964 theorem was on locality and predetermination, and the 1976 theorem on local causality [23]. Whether they are logically fully equivalent is still debated [24]. We have followed closely Einstein's own definition of locality (i.e. his separability criterion) which is different from what appears in the 1935 EPR paper, which was written by Podolsky and which Einstein was not happy with, but which triggered the whole debate. Fairly soon after the paper was published he explained his conceptual position very clearly and succinctly, without using any mathematics, to Schrödinger [25] and later in his ‘Autobiographical Notes’ [15]. We have followed this 1949 version which does not refer to any relativistic space-time, light cones, superluminal signals or causality.

We have also proposed a simple experiment to test the Lüders update postulate.

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